MathAA_11_HL_Summer_2021_Q1

Solution

1. Evaluation of Function Values

- (i) To find $f(2)$, we locate $x = 2$ on the horizontal axis and identify the corresponding y-coordinate on the graph. The point $(2, 6)$ lies on the curve.

$$f(2) = 6$$

- (ii) The expression $(f \circ f)(2)$ represents a **composite function**, defined as $f(f(2))$. Using the result from part (i):

$$\begin{aligned}(f \circ f)(2) &= f(f(2)) \\ &= f(6)\end{aligned}$$

From the provided graph, at $x = 6$, the y-coordinate is -2.

$$(f \circ f)(2) = -2$$

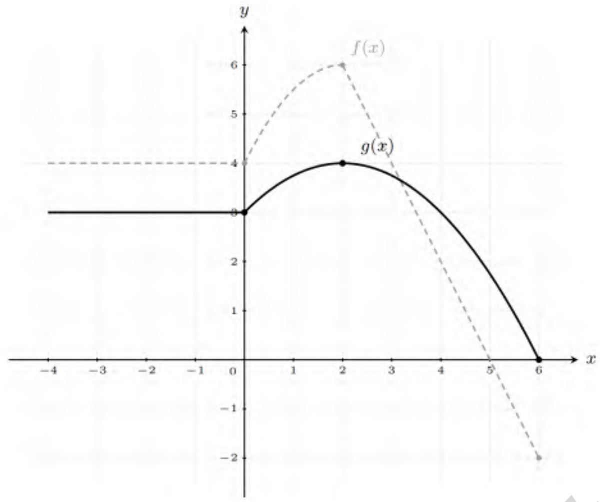
2. Transformation and Sketching of $g(x)$

The function $g(x) = \frac{1}{2}f(x) + 1$ is a transformation of $f(x)$ involving a **vertical stretch** (compression) and a **vertical translation**.

- The factor $\frac{1}{2}$ compresses the graph vertically toward the x-axis by a factor of 0.5.
- The term $+1$ shifts the entire graph upward by 1 unit.

We apply the transformation $y \to \frac{1}{2}y + 1$ to key points from the graph of f:

x	$f(x)$	$g(x) = \frac{1}{2}f(x) + 1$	Point on g
-4	4	$\frac{1}{2}(4) + 1 = 3$	$(-4, 3)$
0	4	$\frac{1}{2}(4) + 1 = 3$	$(0, 3)$
2	6	$\frac{1}{2}(6) + 1 = 4$	$(2, 4)$
5	0	$\frac{1}{2}(0) + 1 = 1$	$(5, 1)$
6	-2	$\frac{1}{2}(-2) + 1 = 0$	$(6, 0)$



Final Answers:

(a)
(b) $\boxed{6}$

(ii) $\boxed{-2}$

(b) The graph of $g(x)$ is a horizontal line at $y = 3$ for $x \in [-4, 0]$ and a compressed, shifted parabola for $x \in [0, 6]$ with a vertex at $(2, 4)$ and an x-intercept at $(6, 0)$.

MathAA_11_HL_Summer_2021_Q2

Solution

To find the first term u_1 and the **common difference** d of the **arithmetic sequence**, we utilize the standard formulas for the n-th term and the sum of the first n terms.

1. Establish the governing equations For an arithmetic sequence, the n-th term u_n is given by:

$$u_n = u_1 + (n-1)d$$

The sum of the first n terms S_n is given by:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

2. Apply the given conditions We are given $u_8 = 8$ and $S_8 = 8$.

- Using the sum formula for $n = 8$:

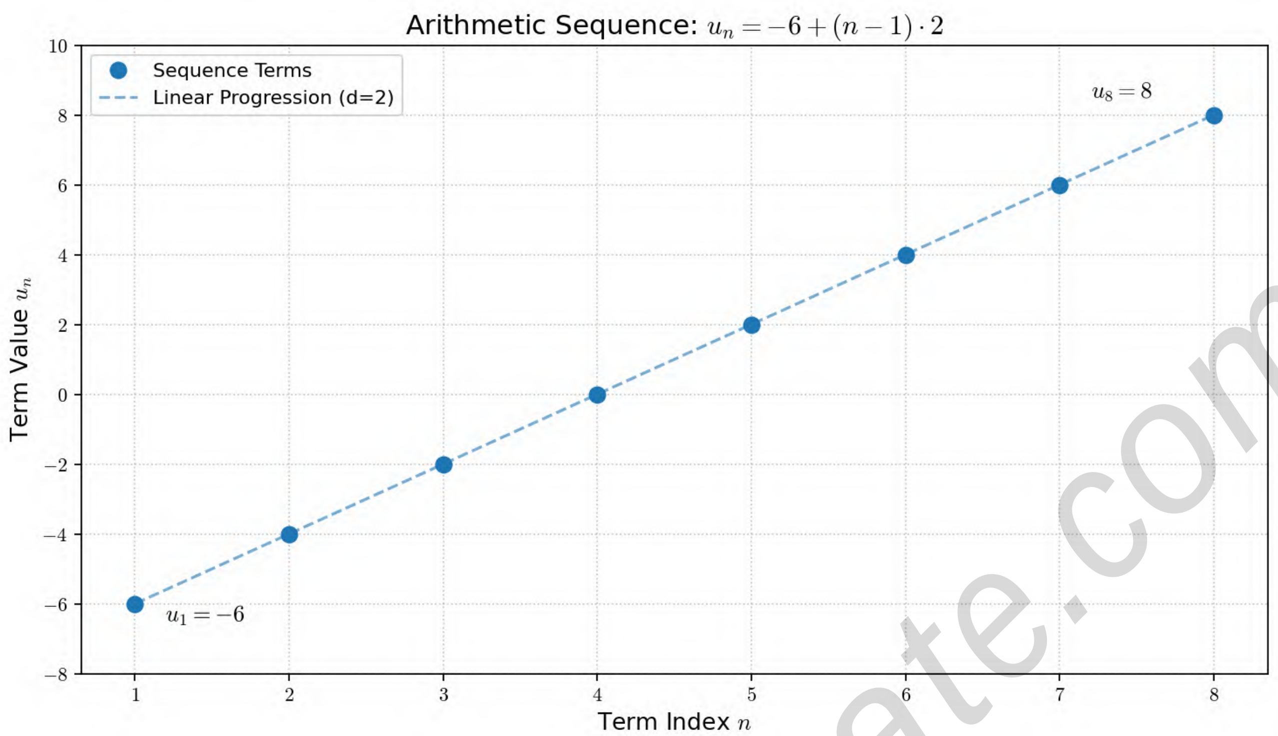
$$\begin{aligned} S_8 &= \frac{8}{2}(u_1 + u_8) \\ 8 &= 4(u_1 + 8) \end{aligned}$$

- Solving for u_1:

$$\begin{aligned} 2 &= u_1 + 8 \\ u_1 &= 2 - 8 \\ u_1 &= -6 \end{aligned}$$

3. Determine the common difference Now, substitute $u_1 = -6$ and $u_8 = 8$ into the n-th term formula for $n = 8$:

$$\begin{aligned} u_8 &= u_1 + (8-1)d \\ 8 &= -6 + 7d \\ 14 &= 7d \\ d &= \frac{14}{7} \\ d &= 2 \end{aligned}$$



4. Verification The sequence is $-6, -4, -2, 0, 2, 4, 6, 8$. The sum $S_8 = -6 - 4 - 2 + 0 + 2 + 4 + 6 + 8 = 8$, which matches the given condition.

$\boxed{u_1 = -6, d = 2}$

MathAA_11_HL_Summer_2021_Q3

Solution

1. Identification of Box Plot Parameters

From the provided **box and whisker diagram**, we identify the following components of the **five-number summary**:

- Minimum value ($x_{\min}$) = $10\,\mathrm{g}$
- Lower quartile (Q_1) = L
- Median (Q_2) = $40\,\mathrm{g}$
- Upper quartile (Q_3) = U
- Maximum value ($x_{\max}$) = $75\,\mathrm{g}$
- **Interquartile range** (IQR) = $20\,\mathrm{g}$

The relationship between the quartiles and the IQR is defined by:

$$IQR = U - L = 20$$

2. Analysis of Outliers

The problem states there are no **outliers** in the results. According to the standard **Tukey's fences** method, a data point is considered an outlier if it lies outside the following boundaries:

- Lower boundary: $L - 1.5 \times IQR$
- Upper boundary: $U + 1.5 \times IQR$

Since there are no outliers, the maximum value ($75\,\mathrm{g}$) must be less than or equal to the upper boundary, and the minimum value ($10\,\mathrm{g}$) must be greater than or equal to the lower boundary.

3. Part (a): Minimum possible value of U

To ensure the maximum value of 75 is not an outlier, it must satisfy:

$$x_{\max} \leq U + 1.5 \times IQR$$

Substituting the known values:

$$\begin{aligned} 75 &\leq U + 1.5(20) \\ 75 &\leq U + 30 \\ U &\geq 75 - 30 \\ U &\geq 45 \end{aligned}$$

Additionally, by the definition of quartiles, the upper quartile U must be greater than or equal to the median (40). Since $45 \geq 40$, the constraint from the outlier rule is the limiting factor. Thus, the minimum possible value for U is 45.

$\boxed{U = 45}$

4. Part (b): Minimum possible value of L

Using the IQR relationship $U - L = 20$, we can express L in terms of U:

$$L = U - 20$$

To find the minimum possible value of L, we must consider the constraints on L.

- From the IQR equation: as U increases, L increases. Therefore, the smallest L occurs when U is at its minimum.
- From the outlier condition for the lower bound:

$$\begin{aligned} x_{\min} &\geq L - 1.5 \times IQR \\ 10 &\geq L - 1.5(20) \\ 10 &\geq L - 30 \\ L &\leq 40 \end{aligned}$$

- Also, L must be less than or equal to the median (40) and greater than or equal to the minimum (10).

Using the result from part (a) where the minimum $U = 45$:

$$\begin{aligned} L_{\min} &= U_{\min} - 20 \\ &= 45 - 20 \\ &= 25 \end{aligned}$$

We check if $L = 25$ satisfies the lower outlier condition: $25 \leq 40$ is true. Thus, the minimum value of L corresponding to the minimum value of U is 25.

$\boxed{L = 25}$

MathAA_11_HL_Summer_2021_Q4

Solution

1. Finding the derivative of $f(x)$

The function $f(x)$ is given by:

$$f(x) = -(x-h)^2 + 2k$$

To find the derivative $f'(x)$, we apply the **power rule** and the **chain rule**:

$$\begin{aligned} f'(x) &= \frac{d}{dx}[-(x-h)^2 + 2k] \\ &= -2(x-h) \cdot \frac{d}{dx}(x-h) + 0 \\ &= -2(x-h) \end{aligned}$$

Thus, the derivative is:

$$\boxed{f'(x) = -2(x-h)}$$

2. Showing that $h = \frac{e+6}{2}$

The problem states that the graphs of f and g have a **common tangent** at $x = 3$. This implies two conditions:

- The functions must have the same value at $x = 3$: $f(3) = g(3)$.
- The functions must have the same gradient at $x = 3$: $f'(3) = g'(3)$.

First, we find the derivative of $g(x) = e^{x-2} + k$:

$$\begin{aligned} g'(x) &= \frac{d}{dx}(e^{x-2} + k) \\ &= e^{x-2} \end{aligned}$$

Equating the gradients at $x = 3$:

$$\begin{aligned} f'(3) &= g'(3) \\ -2(3-h) &= e^{3-2} \\ -6 + 2h &= e^1 \\ 2h &= e + 6 \\ h &= \frac{e+6}{2} \end{aligned}$$

This confirms the required expression for h.

3. Showing that $k = e + \frac{e^2}{4}$

Using the second condition for a common tangent, the function values must be equal at $x = 3$:

$$f(3) = g(3)$$

Substitute the expressions for $f(x)$ and $g(x)$:

$$-(3-h)^2 + 2k = e^{3-2} + k$$

Rearranging to solve for k:

$$k = e^1 + (3-h)^2$$

From the previous step, we know $h = \frac{e+6}{2}$. Substitute this into the equation:

$$\begin{aligned} k &= e + \left(3 - \frac{e+6}{2}\right)^2 \\ &= e + \left(\frac{6-(e+6)}{2}\right)^2 \\ &= e + \left(\frac{-e}{2}\right)^2 \\ &= e + \frac{e^2}{4} \end{aligned}$$

This confirms the required expression for k.

MathAA_11_HL_Summer_2021_Q5

Solution

1. Proof of the Trigonometric Identity

To show that $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$, we apply the **double-angle formulas** for sine and cosine:

- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = 1 - 2\sin^2 x$

Starting with the left-hand side (LHS):

$$\begin{aligned}\text{LHS} &= \sin 2x + \cos 2x - 1 \\ &= (2\sin x\cos x) + (1 - 2\sin^2 x) - 1 \\ &= 2\sin x\cos x - 2\sin^2 x\end{aligned}$$

Factor out the common term $2\sin x$:

$$\text{LHS} = 2\sin x(\cos x - \sin x)$$

This matches the right-hand side (RHS), thus the identity is proven.

2. Solving the Equation

We are given the equation $\sin 2x + \cos 2x - 1 + \cos x - \sin x = 0$ for the interval $0 < x < 2\pi$.

- **Step 1: Substitute the identity from part (a)** Using the result $\sin 2x + \cos 2x - 1 = 2\sin x(\cos x - \sin x)$, the equation becomes:

$$2\sin x(\cos x - \sin x) + (\cos x - \sin x) = 0$$

- **Step 2: Factor the expression** Factor out the common term $(\cos x - \sin x)$:

$$(\cos x - \sin x)(2\sin x + 1) = 0$$

- **Step 3: Solve for each factor** This gives two cases:

  Case 1: $\cos x - \sin x = 0$

$$\begin{aligned}\sin x &= \cos x \\ \tan x &= 1\end{aligned}$$

  In the interval $0 < x < 2\pi$, the solutions for $\tan x = 1$ are:

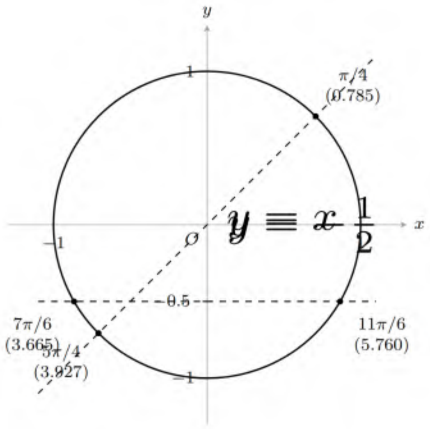
$$x = \frac{\pi}{4}, \quad x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

  Case 2: $2\sin x + 1 = 0$

$$\begin{aligned}2\sin x &= -1 \\ \sin x &= -\frac{1}{2}\end{aligned}$$

  The reference angle is $\alpha = \arcsin(1/2) = \pi/6$. Since sine is negative in the third and fourth quadrants:

$$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}, \quad x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$



- **Step 4: Consolidate solutions** The values of x in the range $0 < x < 2\pi$ are:

$$x = \frac{\pi}{4}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{11\pi}{6}$$

$$\boxed{x = \frac{\pi}{4}, \frac{7\pi}{6}, \frac{5\pi}{4}, \frac{11\pi}{6}}$$

MathAA_11_HL_Summer_2021_Q6

Solution

1. Analysis of the given trigonometric information

The problem provides the value of the **cosecant** function and the interval for the angle θ:

- $\csc\theta = \frac{3}{2}$
- $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

First, we determine the specific **quadrant** in which θ lies. The interval $(\frac{\pi}{2}, \frac{3\pi}{2})$ covers both the second and third quadrants. Since $\csc\theta = \frac{1}{\sin\theta}$ and the given value $\frac{3}{2}$ is positive, $\sin\theta$ must also be positive. In the given interval, the **sine** function is positive only in the second quadrant ($Q2$):

- $\frac{\pi}{2} < \theta < \pi$

2. Application of the Pythagorean identity

To find $\cot\theta$, we utilize the fundamental **Pythagorean identity** relating cosecant and cotangent:

$$1 + \cot^2\theta = \csc^2\theta$$

Substituting the given value:

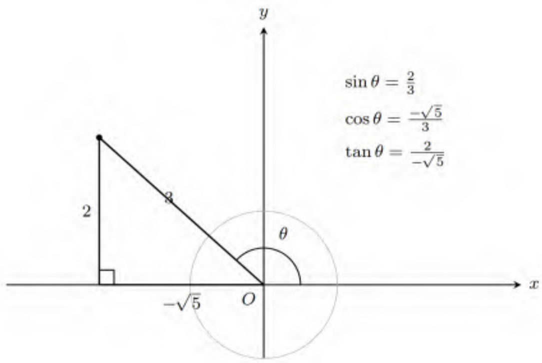
$$\begin{aligned}\cot^2\theta &= \csc^2\theta - 1 \\ &= \left(\frac{3}{2}\right)^2 - 1 \\ &= \frac{9}{4} - 1 \\ &= \frac{5}{4}\end{aligned}$$

3. Determining the sign and final value

Taking the square root of both sides, we have:

$$\cot\theta = \pm\sqrt{\frac{5}{4}} = \pm\frac{\sqrt{5}}{2}$$

As established in step 1, θ is in the second quadrant. In the second quadrant, the **cosine** is negative and the sine is positive, which implies that the **cotangent** ($\frac{\cos\theta}{\sin\theta}$) must be negative.



Therefore, we select the negative root:

$$\cot\theta = -\frac{\sqrt{5}}{2}$$

$$\boxed{-\frac{\sqrt{5}}{2}}$$

MathAA_11_HL_Summer_2021_Q7

Solution

To find the possible values of a for the **quartic equation** $z^4 + 4z^3 + 8z^2 + 80z + 400 = 0$, we utilize the properties of polynomials with real coefficients and **Vieta's formulas**.

1. Identify the roots and their conjugates Since the coefficients of the polynomial are real, any non-real complex root must appear with its **complex conjugate**. We are given two roots:

- $z_1 = a + bi$
- $z_2 = b + ai$ where $a, b \in \mathbb{Z}$.
  - If $a \neq 0$ and $b \neq 0$, and $a \neq b$, then the conjugates $z_3 = \bar{z}_1 = a - bi$ and $z_4 = \bar{z}_2 = b - ai$ must also be roots.
  - If $a = b$, then $z_1 = z_2 = a + ai$. The conjugates are $z_3 = z_4 = a - ai$.
  - If $b = -a$, then $z_1 = a - ai$ and $z_2 = -a + ai$. Their conjugates are $z_3 = a + ai$ and $z_4 = -a - ai$.

In all cases, the set of roots consists of $\{a + bi, a - bi, b + ai, b - ai\}$.

2. Apply Vieta's formulas For the polynomial $P(z) = z^4 + 4z^3 + 8z^2 + 80z + 400$, the sum of the roots and the product of the roots are:

- Sum of roots: $\sum z_i = -4$
- Product of roots: $\prod z_i = 400$

Using the roots identified:

- Sum:

$$\begin{aligned}(a + bi) + (a - bi) + (b + ai) + (b - ai) &= -4 \\ 2a + 2b &= -4 \\ a + b &= -2\end{aligned}$$

- Product:

$$\begin{aligned}(a + bi)(a - bi) \cdot (b + ai)(b - ai) &= 400 \\ (a^2 + b^2) \cdot (b^2 + a^2) &= 400 \\ (a^2 + b^2)^2 &= 400 \\ a^2 + b^2 &= 20\end{aligned}$$

(Note: $a^2 + b^2$ must be positive, so we take the positive square root).

3. Solve the system of equations We have a system of two equations with $a, b \in \mathbb{Z}$:

1. $b = -2 - a$
2. $a^2 + b^2 = 20$

Substitute (1) into (2):

$$
\begin{aligned}
a^2 + (-2-a)^2 &= 20 \\
a^2 + (4 + 4a + a^2) &= 20 \\
2a^2 + 4a + 4 &= 20 \\
2a^2 + 4a - 16 &= 0 \\
a^2 + 2a - 8 &= 0
\end{aligned}
$$

4. Find the integer solutions for a Factor the **quadratic equation**:

$$(a+4)(a-2) = 0$$

The possible values for a are:

- $a = -4$
- $a = 2$

5. Verify the values

- If $a = -4$, then $b = -2 - (-4) = 2$.
  - Roots: $-4 \pm 2i$ and $2 \pm 4i$.
  - Sum: $(-4+2i) + (-4-2i) + (2+4i) + (2-4i) = -8 + 4 = -4$ (Correct).
  - Product: $((-4)^2 + 2^2) \cdot (2^2 + (-4)^2) = 20 \cdot 20 = 400$ (Correct).
- If $a = 2$, then $b = -2 - 2 = -4$.
  - This results in the same set of roots and values for a and b.

The possible values of a are -4 and 2.

$\boxed{a = -4, 2}$

MathAA_11_HL_Summer_2021_Q8

Solution

To evaluate the limit $\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right)$, we follow these steps:

1. Verify the Indeterminate Form First, we check the values of the numerator and denominator as x approaches 0:

- For the numerator: $\lim_{x\to 0}\arctan(2x) = \arctan(0) = 0$.
- For the denominator: $\lim_{x\to 0}\tan(3x) = \tan(0) = 0$.

Since the limit results in the **indeterminate form** $0/0$, we can apply **L'Hôpital's rule**.

2. Apply L'Hôpital's Rule According to L'Hôpital's rule, if $\lim_{x\to c}\frac{f(x)}{g(x)}$ results in $0/0$ or ∞/∞, then:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

- Let $f(x) = \arctan(2x)$. Using the **chain rule** and the derivative of the **arctangent** function $\frac{d}{du}(\arctan u) = \frac{1}{1+u^2}$:

$$f'(x) = \frac{1}{1+(2x)^2}\cdot\frac{d}{dx}(2x) = \frac{2}{1+4x^2}$$

- Let $g(x) = \tan(3x)$. Using the chain rule and the derivative of the **tangent** function $\frac{d}{du}(\tan u) = \sec^2 u$:

$$g'(x) = \sec^2(3x)\cdot\frac{d}{dx}(3x) = 3\sec^2(3x)$$

3. Evaluate the New Limit Now, we substitute these derivatives back into the limit expression:

$$\begin{aligned}\lim_{x\to 0}\frac{\arctan 2x}{\tan 3x} &= \lim_{x\to 0}\frac{\frac{2}{1+4x^2}}{3\sec^2(3x)}\\ &= \lim_{x\to 0}\frac{2}{3(1+4x^2)\sec^2(3x)}\end{aligned}$$

As $x\to 0$:

- $1+4x^2 \to 1+4(0)^2 = 1$
- $\sec^2(3x) = \frac{1}{\cos^2(3x)} \to \frac{1}{\cos^2(0)} = \frac{1}{1^2} = 1$

Substituting these values:

$$\begin{aligned}\lim_{x\to 0}\frac{2}{3(1+4x^2)\sec^2(3x)} &= \frac{2}{3(1)(1)}\\ &= \frac{2}{3}\end{aligned}$$

$\boxed{\frac{2}{3}}$

MathAA_11_HL_Summer_2021_Q9

Solution

1. Analysis of Case (a)

In this scenario, there are $n = 6$ pens and $k = 5$ distinct sheep (Amber, Brownie, Curly, Daisy, and Eden). Each pen can hold any number of sheep. We must find the number of ways to distribute the sheep such that Amber (A) and Brownie (B) are not in the same pen.

- **Total distributions without restrictions:** Since each of the 5 sheep can be placed in any of the 6 pens independently, the total number of ways is:

$$N_{\text{total}} = 6^5$$

- **Distributions where Amber and Brownie are in the same pen:** If A and B are treated as a single unit (since they must be in the same pen), we effectively have 4 "entities" to place (the AB pair, C, D, and E). Each entity has 6 choices of pens:

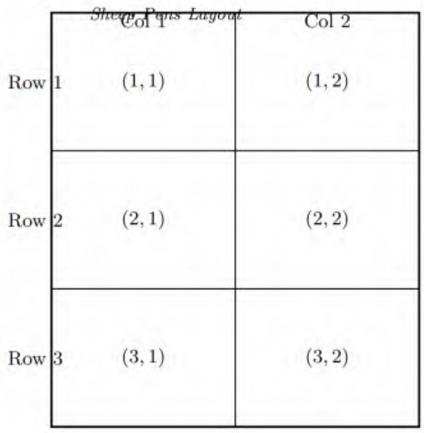
$$N_{\text{same}} = 6^4$$

- **Applying the** Complementary Counting** principle:** The number of valid ways is the difference between the total ways and the restricted ways:

$$\begin{aligned} N_a &= 6^5 - 6^4 \\ &= 7776 - 1296 \\ &= 6480 \end{aligned}$$

2. Analysis of Case (b)

In this scenario, each pen can contain at most one sheep. This is a **Permutation** problem with adjacency constraints. We first place Amber (A) and Brownie (B) such that they do not share a boundary, then place the remaining 3 sheep in the remaining 4 pens.



- **Step 1: Placing Amber and Brownie** We count the pairs of pens that share a boundary.
  - Horizontal boundaries: 3 (one per row)
  - Vertical boundaries: 4 (two per column)
  - Total adjacent pairs: $3 + 4 = 7$
  - Since A and B are distinct, there are $7 \times 2 = 14$ ways they could be placed in adjacent pens.
  - Total ways to place A and B in any two distinct pens is $P(6, 2) = 6 \times 5 = 30$.
  - Ways to place A and B such that they do **not** share a boundary:

$$30 - 14 = 16$$

- **Step 2: Placing the remaining sheep** After A and B are placed, there are $6 - 2 = 4$ pens remaining for the 3 remaining sheep (C, D, E). The number of ways to place them is:

$$P(4, 3) = 4 \times 3 \times 2 = 24$$

- **Step 3: Total calculation** Using the **Fundamental Counting Principle**:

$$\begin{aligned} N_b &= 16 \times 24 \\ &= 384 \end{aligned}$$

Final Answers:

(a) $\boxed{6480}$

(b) $\boxed{384}$

MathAA_11_HL_Summer_2021_Q10

Solution

1. Determination of the parameter p

For any **discrete probability distribution**, the sum of all probabilities must equal 1. From the table for die A:

$$\begin{aligned} \sum P(X=x) = p + p + p + \frac{1}{2}p = 1 \\ 3p + 0.5p = 1 \\ 3.5p = 1 \\ p = \frac{1}{3.5} = \frac{2}{7} \end{aligned}$$

$$\boxed{p = \frac{2}{7}}$$

2. Calculation of the expected value $E(X)$

The **expected value** is defined as $E(X) = \sum x \cdot P(X = x)$.

$$\begin{aligned} E(X) &= 1(p) + 2(p) + 3(p) + 4\left(\frac{1}{2}p\right) \\ &= p + 2p + 3p + 2p \\ &= 8p \end{aligned}$$

Substituting $p = \frac{2}{7}$:

$$E(X) = 8 \cdot \frac{2}{7} = \frac{16}{7}$$

$$\boxed{E(X) = \frac{16}{7}}$$

3. Range of possible values for r and q

- **(i) Range of r:** Since r is a probability, it must satisfy $0 \leq r \leq 1$. However, for the other probabilities q to exist and be valid probabilities, we must consider the constraint $\sum P(Y = y) = 1$:

  $$3q + r = 1 \implies q = \frac{1-r}{3}$$

  For q to be a valid probability, $q \geq 0$, which implies $r \leq 1$. If we assume the die must be able to land on all faces (or at least that q is non-negative), the standard range for a single probability is: $\boxed{0 \leq r \leq 1}$

- **(ii) Range of q:** Using the relation $3q + r = 1$, we solve for q based on the bounds of r:
  - If $r = 0$, $3q = 1 \implies q = \frac{1}{3}$.
  - If $r = 1$, $3q = 0 \implies q = 0$. Thus, the range for q is: $\boxed{0 \leq q \leq \frac{1}{3}}$

4. Range of possible values for $E(Y)$

The expected value for die B is:

$$E(Y) = 1(q) + 2(q) + 3(q) + 4(r)$$
$$= 6q + 4r$$

Substitute $r = 1 - 3q$:

$$E(Y) = 6q + 4(1 - 3q)$$
$$= 6q + 4 - 12q$$
$$= 4 - 6q$$

Using the range $0 \le q \le \frac{1}{3}$: - If $q = 0$, $E(Y) = 4 - 6(0) = 4$. - If $q = \frac{1}{3}$, $E(Y) = 4 - 6\left(\frac{1}{3}\right) = 4 - 2 = 2$. $\boxed{2 \le E(Y) \le 4}$

5. Determination of $E(Y)$ given $P(X < Y) = \frac{1}{2}$

We calculate the **joint probability** $P(X < Y)$ by summing the probabilities of all pairs (x, y) such that $x < y$. Since the rolls are **independent events**, $P(X = x, Y = y) = P(X = x)P(Y = y)$.

$$P(X < Y) = P(X = 1)[P(Y = 2) + P(Y = 3) + P(Y = 4)]$$
$$+ P(X = 2)[P(Y = 3) + P(Y = 4)]$$
$$+ P(X = 3)[P(Y = 4)]$$

Substitute the probabilities: $P(X = 1) = p, P(X = 2) = p, P(X = 3) = p$ and $P(Y = 1) = q, P(Y = 2) = q, P(Y = 3) = q, P(Y = 4) = r$.

$$\begin{aligned}P(X<Y) &= p(q+q+r)+p(q+r)+p(r)\\ &= p(2q+r)+p(q+r)+pr\\ &= p(2q+r+q+r+r)\\ &= p(3q+3r)\end{aligned}$$

Given $P(X<Y)=\frac{1}{2}$ and $p=\frac{2}{7}$:

$$\begin{aligned}\frac{2}{7}(3q+3r) &= \frac{1}{2}\\ 3q+3r &= \frac{7}{4}\\ q+r &= \frac{7}{12}\end{aligned}$$

We also know the normalization condition for die B: $3q+r=1$. We now have a system of linear equations: 1) $q+r=\frac{7}{12}$ 2) $3q+r=1$

Subtract (1) from (2):

$$\begin{aligned}(3q+r)-(q+r) &= 1-\frac{7}{12}\\ 2q &= \frac{5}{12}\\ q &= \frac{5}{24}\end{aligned}$$

Now find $E(Y)$ using the expression derived in step 4:

$$\begin{aligned}E(Y) &= 4-6q\\ &= 4-6\left(\frac{5}{24}\right)\\ &= 4-\frac{5}{4}\\ &= \frac{16-5}{4}=\frac{11}{4}\end{aligned}$$

$\boxed{E(Y)=2.75}$

MathAA_11_HL_Summer_2021_Q11

Solution

1. Analysis of Line L_1

- **(i) Verification of point on L_1** The **Cartesian equation** of line L_1 is given by:

$$\frac{x+1}{2} = y = 3 - z$$

  To show that the point $(-1, 0, 3)$ lies on L_1, we substitute $x = -1$, $y = 0$, and $z = 3$ into the equation:

$$\frac{-1+1}{2} = 0$$
$$y = 0$$
$$3 - 3 = 0$$

  Since $0 = 0 = 0$, the point $(-1, 0, 3)$ satisfies the equation and thus lies on L_1.

- **(ii) Vector equation of L_1** Let $\frac{x+1}{2} = y = 3 - z = \lambda$. We can express x, y, z in terms of the parameter λ:

$$x = 2\lambda - 1$$
$$y = \lambda$$
$$z = 3 - \lambda$$

  In vector form, $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. The **vector equation** is:

$$\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$$

2. Angle between L_1 and L_2

The direction vectors of L_1 and L_2 are $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix}$ respectively. The **acute angle** θ between two lines is given by the **dot product** formula:

$$\cos\theta = \frac{|\ \mathbf{v}_1 \cdot \mathbf{v}_2\ |}{|\ \mathbf{v}_1\ |\ |\ \mathbf{v}_2\ |}$$

Given $\theta = 45^\circ$, then $\cos 45^\circ = \frac{1}{\sqrt{2}}$. - $\mathbf{v}_1 \cdot \mathbf{v}_2 = (2)(a) + (1)(1) + (-1)(-1) = 2a + 2$ - $|\ \mathbf{v}_1\ | = \sqrt{2^2 + 1^2 + (-1)^2} = \sqrt{6}$ - $|\ \mathbf{v}_2\ | = \sqrt{a^2 + 1^2 + (-1)^2} = \sqrt{a^2 + 2}$

Substituting these into the formula:

$$\frac{1}{\sqrt{2}} = \frac{|\ 2a+2\ |}{\sqrt{6}\sqrt{a^2+2}}$$
$$\sqrt{3}\sqrt{a^2+2} = |\ 2a+2\ |$$
$$3(a^2+2) = (2a+2)^2$$
$$3a^2+6 = 4a^2+8a+4$$
$$a^2+8a-2 = 0$$

Using the **quadratic formula** $a = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$:

$$a = \frac{-8 \pm \sqrt{64-4(1)(-2)}}{2}$$
$$= \frac{-8 \pm \sqrt{72}}{2}$$
$$= \frac{-8 \pm 6\sqrt{2}}{2}$$
$$= -4 \pm 3\sqrt{2}$$

$\boxed{a = -4+3\sqrt{2}, -4-3\sqrt{2}}$

3. Intersection of L_1 and L_2

The lines are: $L_1 : \mathbf{r} = \begin{pmatrix} -1+2\lambda \\ \lambda \\ 3-\lambda \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} at \\ 1+t \\ 2-t \end{pmatrix}$ For intersection, equate the components: 1) $-1+2\lambda = at$ 2) $\lambda = 1+t$ 3) $3-\lambda = 2-t$

From (2), $\lambda - t = 1$. From (3), $3-2 = \lambda - t \implies 1 = \lambda - t$. Equations (2) and (3) are consistent and dependent. Substitute $\lambda = 1+t$ into (1):

$$-1+2(1+t) = at$$
$$-1+2+2t = at$$
$$1+2t = at$$
$$t(a-2) = 1$$

For a unique point of intersection, $a-2 \neq 0$, so $a \neq 2$. Thus, $k=2$. The parameter t is:

$$t = \frac{1}{a-2}$$

Substitute t into the equation for L_2 to find the coordinates of A:

$$x_A = a\left(\frac{1}{a-2}\right) = \frac{a}{a-2}$$
$$y_A = 1 + \frac{1}{a-2} = \frac{a-2+1}{a-2} = \frac{a-1}{a-2}$$
$$z_A = 2 - \frac{1}{a-2} = \frac{2a-4-1}{a-2} = \frac{2a-5}{a-2}$$

$\boxed{k = 2, A = \left(\frac{a}{a-2}, \frac{a-1}{a-2}, \frac{2a-5}{a-2}\right)}$

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Solution

1. Derivation of the second derivative

To find $f''(x)$ for $f(x) = \sqrt{1+x} = (1+x)^{1/2}$, we apply the **power rule** for differentiation:

- First derivative:

$$\begin{aligned} f'(x) &= \frac{1}{2}(1+x)^{1/2-1} \\ &= \frac{1}{2}(1+x)^{-1/2} \end{aligned}$$

- Second derivative:

$$\begin{aligned} f''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-1/2-1} \\ &= -\frac{1}{4}(1+x)^{-3/2} \\ &= -\frac{1}{4\sqrt{(1+x)^3}} \end{aligned}$$

2. Proof by Mathematical Induction

We wish to prove that for $n \in \mathbb{Z}, n \geq 2$:

$$f^{(n)}(x) = \left(-\frac{1}{4}\right)^{n-1} \frac{(2n-3)!}{(n-2)!}(1+x)^{\frac{1}{2}-n}$$

- **Base Case:** For $n = 2$:
  - LHS: $f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ (from part a).
  - RHS: $\left(-\frac{1}{4}\right)^{2-1} \frac{(2(2)-3)!}{(2-2)!}(1+x)^{\frac{1}{2}-2} = \left(-\frac{1}{4}\right)^{1} \frac{1!}{0!}(1+x)^{-3/2} = -\frac{1}{4}(1+x)^{-3/2}$. The base case holds.
- **Inductive Step:** Assume the formula holds for $n = k$:

$$f^{(k)}(x) = \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}$$

  Differentiate both sides with respect to x to find $f^{(k+1)}(x)$:

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left[\left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}(1+x)^{\frac{1}{2}-k}\right] \\ &= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1}{2}-k\right)(1+x)^{\frac{1}{2}-k-1} \\ &= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(\frac{1-2k}{2}\right)(1+x)^{\frac{1}{2}-(k+1)} \\ &= \left(-\frac{1}{4}\right)^{k-1} \frac{(2k-3)!}{(k-2)!}\left(-\frac{2k-1}{2}\right)(1+x)^{\frac{1}{2}-(k+1)} \end{aligned}$$

  To match the form for $n = k+1$, we manipulate the constants:

$$f^{(k+1)}(x) = \left(-\frac{1}{4}\right)^{k-1}\left(-\frac{1}{4}\cdot 2\right)\frac{(2k-3)!}{(k-2)!}(2k-1)(1+x)^{\frac{1}{2}-(k+1)}$$
$$= \left(-\frac{1}{4}\right)^{k}\frac{(2k-3)!(2k-1)(2k-2)}{(k-2)!(k-1)}(1+x)^{\frac{1}{2}-(k+1)}\cdot\frac{1}{2}\cdot 2$$
$$= \left(-\frac{1}{4}\right)^{k}\frac{(2k-1)!}{(k-1)!}(1+x)^{\frac{1}{2}-(k+1)}$$

Substituting $n = k+1$ into the original formula gives $\frac{(2(k+1)-3)!}{((k+1)-2)!} = \frac{(2k-1)!}{(k-1)!}$. Thus, the inductive step is complete. By the principle of **mathematical induction**, the formula is true for all $n \geq 2$.

3. Finding the values of m

We consider the **Maclaurin series** for $h(x) = f(x)g(x) = \sqrt{1+x}\cdot e^{mx}$.

- The expansion for $f(x) = (1+x)^{1/2}$ is:

$$f(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \ldots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$$

- The expansion for $g(x) = e^{mx}$ is:

$$g(x) = 1 + mx + \frac{(mx)^2}{2!} + \ldots = 1 + mx + \frac{m^2}{2}x^2 + \ldots$$

- Multiplying the series to find the x^2 term of $h(x)$:

$$h(x) = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots\right)\left(1 + mx + \frac{m^2}{2}x^2 + \ldots\right)$$

$$\text{Coeff of } x^2 = (1)\left(\frac{m^2}{2}\right) + \left(\frac{1}{2}\right)(m) + \left(-\frac{1}{8}\right)(1)$$
$$= \frac{m^2}{2} + \frac{m}{2} - \frac{1}{8}$$

Given the coefficient is $\frac{7}{4}$:

$$\frac{m^2}{2} + \frac{m}{2} - \frac{1}{8} = \frac{7}{4}$$
$$4m^2 + 4m - 1 = 14$$
$$4m^2 + 4m - 15 = 0$$

Solving the **quadratic equation** using the quadratic formula:

$$m = \frac{-4 \pm \sqrt{4^2 - 4(4)(-15)}}{2(4)}$$
$$= \frac{-4 \pm \sqrt{16+240}}{8}$$
$$= \frac{-4 \pm \sqrt{256}}{8}$$
$$= \frac{-4 \pm 16}{8}$$

This yields two possible values:

- $m_1 = \frac{12}{8} = \frac{3}{2}$
- $m_2 = \frac{-20}{8} = -\frac{5}{2}$

$$\boxed{m = \frac{3}{2}, -\frac{5}{2}}$$